

A note on the torque anomaly

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I reproduce, in the case of a conical geometry, the torque anomaly recently noted by Fulling, Mera and Trendafilova for the wedge. The expected conservation equation is obtained by a variational method and a mathematical cancellation of the anomaly is exhibited, motivated by the process of truncating the cone at some inner radius.

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1. Introduction

Interest in quantum field theory in a spatial wedge has remained at a certain level for several years and recently a possible anomaly concerning the vacuum averaged energy momentum tensor has surfaced, [1]. Before describing this, I wish to revisit my old wedge, and cone, calculations, some of which remain unpublished, even though various other treatments have appeared in the meantime.

In fact, the earliest analysis in the wedge geometry is the often neglected pioneer work of Lukosz, [2] who makes most of the relevant points, including the importance of the asymptotic distribution of eigenvalues and the various cancellations that can occur. He is concerned with the vacuum energy of electromagnetic field which he reduces to the scalar field problem (plus some additions).

2. Vacuum averages in a wedge

I begin with some old things by calculating $\langle T_{00} \rangle$ in a wedge. Using the coincidence limit expression, the improved energy momentum tensor average can be written in the present case as

$$\langle T_{00} \rangle(x) = \frac{i}{6} \lim_{x' \rightarrow x} (5\partial_t \partial_t - \partial_i \partial_{i'}) D(x, x') \quad (1)$$

where the (Feynman) Green function in the wedge of angle β is, ²

$$\begin{aligned} D(x, x') &= \frac{i}{8\pi\beta r r' \sinh \alpha_1} \left(\frac{\sinh(\pi\alpha_1)/\beta}{\cosh(\pi\alpha_1/\beta) - \cos(\pi(\phi - \phi')/\beta)} \mp (\phi' \rightarrow -\phi') \right) \\ &\equiv D^+ \mp D^- \end{aligned} \quad (2)$$

for D or N conditions on the sides of the wedge, $\phi = 0, \phi = \beta$. The coordinates on the wedge are (t, r, ϕ, z) and,

$$\cosh \alpha_1 = \frac{r^2 + r'^2 + (z - z')^2 - (t - t')^2}{2r r'}. \quad (3)$$

Equation (2) should be interpreted distributionally with the appropriate $i\epsilon$.

I remark now that a subtraction will probably be necessary. This is done later.

² This result is derived by Lukosz on the basis of images when $\beta = \pi/m$, $m \in \mathbb{Z}$, and then checking for any β . It follows more simply from the complex contour analysis of Carslaw, [3], as a residue.

The first part, D^+ , I refer to as the direct term. When images work, it contains those obtained through an even number of reflections, while D^- , the indirect term, contains the images coming from an odd number. For D conditions, these are the negative images. When $\beta = \pi$, the direct term is the usual Minkowski Feynman Green function.

I now compute the coincidence limits in (1), one by one. The calculation is similar to that in [4]. My procedure is to set ϕ equal to ϕ' as the very last step. In the intermediate stages since there are no infinities I can choose whatever method and order of coincidences that is most convenient.³

The t and z cases are similar and proceed via differentiation with respect to α_1 using,

$$\frac{\partial}{\partial t} = \frac{\partial \alpha_1}{\partial t} \frac{\partial}{\partial \alpha_1}, \quad \frac{\partial}{\partial z} = \frac{\partial \alpha_1}{\partial z} \frac{\partial}{\partial \alpha_1}$$

with

$$\sinh \alpha_1 \frac{\partial \alpha_1}{\partial t} = -\frac{t - t'}{r r'}, \quad \sinh \alpha_1 \frac{\partial \alpha_1}{\partial z} = \frac{z - z'}{r r'}.$$

The second derivatives are needed. I spell things out,

$$\begin{aligned} \partial_t \partial_t &= \partial_t ((\partial_t \alpha_1) \partial_{\alpha_1}) \\ &= \partial_t \left(-\frac{t - t'}{r r' \sinh \alpha_1} \right) \partial_{\alpha_1} + (\partial_t \alpha_1)^2 \partial_{\alpha_1}^2 \\ &= \frac{t - t'}{r r'} \frac{\cosh \alpha_1}{\sinh^2 \alpha_1} \partial_t \alpha_1 \partial_{\alpha_1} - \frac{1}{r r' \sinh \alpha_1} \partial_{\alpha_1} + (\partial_t \alpha_1)^2 \partial_{\alpha_1}^2 \end{aligned}$$

I now set $t = t'$ keeping α_1 non zero, as I may, so that $\partial_t \alpha_1 = 0$ and⁴

$$\lim \partial_t \partial_t = -\lim \frac{1}{r r' \sinh \alpha_1} \partial_{\alpha_1} \Big|_{\alpha_1 \rightarrow 0} = -\frac{1}{r^2} \partial_{\alpha_1}^2 \Big|_{\alpha_1=0}, \quad (4)$$

(still with $\phi \rightarrow \phi'$ at the very end). $\alpha_1 = 0$ entails $z = z'$ and $r = r'$. The z coordinate gives likewise,

$$\lim \partial_z \partial_{z'} = -\frac{1}{r^2} \partial_{\alpha_1}^2 \Big|_{\alpha_1=0}. \quad (5)$$

There are three possible terms for the necessary r derivatives arising from the dependence of D on r and r' through (a) α_1 , (b) the external factor $1/r r'$ and (c) a

³ Some queries regarding the point splitting procedure when boundaries are present have been lately raised.

⁴ Inspection of (2) shows that D is even in α_1 and for small α_1 looks like $a + b\alpha_1^2 + \dots$

cross term involving both. Simplification results on immediately setting $t = t'$ and $z = z'$ when,

$$\cosh \alpha_1 = \frac{r^2 + r'^2}{2r r'} , \quad \sinh \alpha_1 = \pm \frac{r^2 - r'^2}{2r r'}$$

and so

$$\partial_r = (\partial_r \alpha_1) \partial_{\alpha_1}$$

with

$$\cosh \alpha_1 \partial_r \alpha_1 = \pm \frac{r^2 + r'^2}{2r^2 r'} ,$$

whence. to begin with,

$$\partial_r \alpha_1 = \pm \frac{1}{r} .$$

Similarly,

$$\partial_{r'} \alpha_1 = \mp \frac{1}{r} .$$

For the second derivative, I compute first the contribution from just the dependence on α_1 ,

$$\begin{aligned} \partial_{r'} \partial_r &= \partial_{r'} (\partial_r \alpha_1 \partial_{\alpha_1}) \\ &= \partial_{r'} (\partial_r \alpha_1) \cdot \partial_{\alpha_1} + \partial_r \alpha_1 \cdot \partial_{r'} \partial_{\alpha_1} \\ &= \partial_{r'} (1/r) \cdot \partial_{\alpha_1} + \partial_r \alpha_1 \partial_{r'} \alpha_1 \cdot \partial^2 \alpha_1 \\ &= 0 - \frac{1}{r r'} \partial^2 \alpha_1 , \end{aligned}$$

and so for the limit,

$$\lim \partial_{r'} \partial_r = -\frac{1}{r^2} \partial^2 \alpha_1 \Big|_{\alpha_1=0} , \quad (6)$$

which is the same as the t and z results, (4), (5). To this must be added the result of acting on the external r, r' factors in the Green function (2). It is easily seen that this yields the contribution in the limit,

$$\lim (\partial_r \partial_{r'} D) \Big|_{ext} = \frac{i}{8\beta^2 r^4} \left(\frac{1}{1 - \cos \Phi^-} \mp (\Phi^- \rightarrow \Phi^+) \right) , \quad (7)$$

where, for short, I have introduced $\Phi^\mp = \pi(\phi \mp \phi')/\beta$.

There is yet a third contribution coming from one r derivative acting on the external factors and the other on the α_1 dependence. This however vanishes in the limit $\alpha_1 \rightarrow 0$ because it involves the *first* derivative of D with respect to α_1 , and this is zero at $\alpha_1 = 0$ since D is even.

At this stage I rewrite the desired vacuum average, (1) in the light of these computations in the intermediately reduced form,

$$\langle T_{00} \rangle = \frac{i}{6} \lim \left(\left(3\partial_t \partial_t - \frac{1}{r^2} \partial_\phi \partial_{\phi'} \right) D - (\partial_r \partial_{r'}) D \right) \Big|_{ext}. \quad (8)$$

The angular derivations are now wanted. Doing these directly, the first one is,

$$\partial_\phi D = -\frac{i}{8\beta^2 r r' \sinh \alpha_1} \left(\frac{\sinh(\pi \alpha_1 / \beta) \sin \Phi^-}{(\cosh(\pi \alpha_1 / \beta) - \cos \Phi^-)^2} \mp (\Phi^- \rightarrow \Phi^+) \right),$$

while for the second one, for brevity, I simultaneously take the limit, $\alpha_1 \rightarrow 0$,

$$\lim \partial_\phi \partial_{\phi'} D = -\frac{i\pi^2}{8\beta^4 r^2} \left(\frac{2 + \cos \Phi^-}{(1 - \cos \Phi^-)^2} \pm (\Phi^- \rightarrow \Phi^+) \right).$$

The last two terms in the sought for quantity (8) are now explicit. To find the first term the second derivative of the Green function (2) with respect to α_1 at $\alpha_1 = 0$ is needed, according to (4), and this can be found by expansion, [4]. I get,

$$3 \lim \partial_t \partial_t D = \frac{i}{8\beta^2 r^4} \left(\frac{1}{1 - \cos \Phi^-} + \frac{\pi^2}{\beta^2} \frac{2 + \cos \Phi^-}{(1 - \cos \Phi^-)^2} \mp (\Phi^- \rightarrow \Phi^+) \right),$$

and all that remains is to combine these expressions according to (8), address the subtraction question and take the final limit of $\phi' \rightarrow \phi$.

It is easily checked that the indirect terms all cancel, making $\langle T_{00} \rangle$ independent of angle. This is, perhaps, not so surprising, seeing that this average is independent of position (for conformal coupling) in the case of the Casimir parallel plates, an old result, [5].

Adding the direct terms gives,

$$\langle T_{00} \rangle = -\frac{\pi^2}{24\beta^4 r^4} \lim_{\Phi^- \rightarrow 0} \frac{2 + \cos \Phi^-}{(1 - \cos \Phi^-)^2},$$

which limit has a UV divergence cured by the subtraction of the Minkowski expression corresponding to the value $\beta = \pi$. Making this ‘renormalisation’ yields the result quoted in [6] (and computed in 1976),⁵

$$\langle T_{00} \rangle \Big|_{wedge} = -\frac{1}{1440\beta^2 r^4} \left(\frac{\pi^2}{\beta^2} - \frac{\beta^2}{\pi^2} \right). \quad (9)$$

⁵ Lukosz did not take the evaluation far enough and his expression still diverges as the cutoff is removed.

Assuming conservation and tracelessness the full $\langle T \rangle_{\mu\nu}$ follows as,

$$\langle T \rangle_{\mu}^{\nu} = \langle T_{00} \rangle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (10)$$

which can be confirmed by direct evaluation using coincidence limits, *e.g.* Christensen, [7], as used by Deutsch and Candelas, [8] and many others.

Deutsch and Candelas, [8], appear to obtain the same result as (9) in their equation (5.42). However a glance at their previous equation shows that they are really calculating for the *sum* of the N and D scalar problems. In this case, as is obvious from the form of the Green function, (2), the indirect terms cancel at the outset in an obvious way and the resulting average is really that appropriate for a *cone*, or cosmic string of periodicity 2β , the angular independence being guaranteed by rotational symmetry. My result, (9), is for a true wedge with D *or* N conditions on the sides, and the cancellation in the average is more subtle and requires conformal invariance.

3. The possible anomaly

In [1] it is shown there appears to be a violation of global energy conservation as the angle of the wedge is altered, the change in the vacuum energy not equaling the work done by the torque on the sides.⁶ A similar difficulty was found, but not resolved, some time ago by the present author in the context of the field theory in the conical geometry around a cosmic string, say. In this case there are no boundaries and the work done by the torque should be thought of as distributed throughout the manifold much as in an expanding soap bubble. This is the same situation as occurs in the field theory on the compact Einstein universe and was referred to in such terms in [9], at finite temperature.

Because the situation is a little simpler I will consider the conical situation. Adding the N and D expressions (which are the same here) on the wedge for the vacuum averaged stress tensor gives that around a cone of periodicity 2β . I therefore, from hereon, *redefine* β by a factor of two to make this periodicity β . Then the (standard) formula is,

$$\langle T_{00} \rangle = -\frac{1}{180\beta^2 r^4} \left(\frac{4\pi^2}{\beta^2} - \frac{\beta^2}{4\pi^2} \right), \quad (11)$$

⁶ It is interesting that Lukosz already remarks on the necessity of this equality.

still with (10).

The essential part of the metric is $-(dr^2 + r^2 d\phi^2)$ with the range $0 \leq \phi \leq \beta$ or,

$$-\left(dr^2 + \left(\frac{\beta}{2\pi}\right)^2 r^2 d\bar{\phi}^2\right), \quad 0 \leq \bar{\phi} \leq 2\pi,$$

in terms of the ‘physical’ angle $\bar{\phi} = 2\pi\phi/\beta$ which has fixed limits but a variable weight while ϕ has fixed weight but variable limits. Generally I write,

$$g_{rr}dr^2 + g_{\phi\phi}d\phi^2 = g_{rr}dr^2 + g_{\bar{\phi}\bar{\phi}}d\bar{\phi}^2,$$

with $\sqrt{-g^3}$ equalling r on the left side and $r\beta/2\pi$ on the right.

The variational formulae for $T_{\phi\phi}$ and $T_{\bar{\phi}\bar{\phi}}$ are *formally* the same, *i.e.* ,

$$\langle T_{**} \rangle = \frac{2}{\sqrt{-g}} \frac{\delta L^{(1)}}{\delta g^{**}},$$

in terms of the effective Lagrangian, $L^{(1)}$. This is equivalent to the variation,

$$\begin{aligned} \delta L^{(1)} &= \int d^3x \frac{\delta L^{(1)}}{\delta g^{**}} \delta g^{**} \\ &= \frac{1}{2} \int d^3V g_{00}^{1/2} \langle T_{**} \rangle \delta g^{**}, \end{aligned}$$

where $d^3V = \sqrt{-g^3} d^3x$ is the invariant 3-volume element and, here, $g_{00} = 1$.

I will use $\bar{\phi}$ and relate the variation in $g^{\bar{\phi}\bar{\phi}}$ to variation in the cone angle, β . Thus,

$$\delta g^{\bar{\phi}\bar{\phi}} = \frac{4\pi^2}{\beta^3} \frac{2}{r^2} \delta\beta = -g^{\bar{\phi}\bar{\phi}} \frac{2}{\beta} \delta\beta$$

whence

$$\begin{aligned} \frac{\delta L^{(1)}}{\delta\beta} &= -\frac{1}{2\pi} \int_0^\infty r dr \int_0^{2\pi} d\bar{\phi} \int_{-\infty}^\infty dz \langle T_{\bar{\phi}\bar{\phi}} \rangle g^{\bar{\phi}\bar{\phi}} \\ &= -\frac{1}{2\pi} \int_0^\infty r dr \int_0^{2\pi} d\bar{\phi} \int_{-\infty}^\infty dz \langle T_{\bar{\phi}}^{\bar{\phi}} \rangle. \end{aligned}$$

Since, from symmetry, $\langle T_{\bar{\phi}}^{\bar{\phi}} \rangle$ is independent of z , and also of ϕ , one finds,

$$\frac{\delta L^{(1)}}{\delta\beta} = - \int_0^\infty r dr \langle T_{\bar{\phi}}^{\bar{\phi}} \rangle,$$

where $L^{(1)}$ now stands for the effective Lagrangian per unit z -slice.

Now in a static space–time, the effective Lagrangian equals minus the internal energy, E , here the vacuum energy as we are at zero temperature, [6], so I finally arrive at,

$$\delta E(\beta) = \int_0^\infty r \delta\beta \, dr \, \langle T_{\bar{\phi}}^{\bar{\phi}} \rangle = \int_0^\infty r \delta\beta \, dr \, \langle T_{\phi}^{\phi} \rangle, \quad (12)$$

which is a formal consequence of the principle of virtual work since the integrand can be construed as the total work done, over the entire cone, by the thrust $\langle T_{\phi}^{\phi} \rangle \, dr$ acting over a distance $r \delta\beta$, as in a soap bubble. (The angle ϕ is visually the best to use, as varying β corresponds to changing the extent, or range, of this coordinate, similar to the true boundary case.)

Equation (12) is written down in [1] (for the wedge). The discussion here provides a little more formal backing.

There is one obvious problem with (12). Naive interpretation, using (10), makes the right hand side diverge due to the behaviour at the apex of the cone, $r = 0$, and so does the left–hand side if we believe, as seems reasonable, that E is obtained by integrating the energy density, (11). Hence (12) says nothing. However, if the apex divergence is artificially smoothed, say by introducing a suitable test function, $f(r)$, or, more crudely, cut out by just integrating from $r_0 > 0$, then it is impossible to satisfy (12), with the formula,

$$E = \beta \int_{r_0}^\infty r \, dr \, \langle T_{00} \rangle,$$

and (11), with the relation (10).

The discrepancy alluded to earlier amounts to a mismatch between the derivative of the function of β in (9) and the function itself according to (12). I will not write out the details as the algebra is more or less identical to that displayed in [1] where the same attitude to the apex divergence is taken also I exhibit an artificial elimination of the obstruction in the next section.

4. Synthetic cancellation of mismatch

I introduce yet another symbol and define $B = 2\pi/\beta$. Then

$$\langle T_{00} \rangle = \frac{1}{720\pi^2 r^4} P_4(B)$$

where the polynomial is defined by

$$P_n(B) = B^n - 1.$$

The lower polynomial P_2 enters into the expression for the vacuum average of the square of the field, $\langle \varphi^2 \rangle$, which motivates the following.

I make an ansatz for the truncated total energy,

$$E(B, r_0) \equiv \frac{2\pi}{B} \int_{r_0}^{\infty} r dr \langle T_{00} \rangle ,$$

viz,

$$E(B, r_0) = N(r_0) \frac{P_4(B)}{B} + M(r_0) \frac{P_2(B)}{B} , \quad (13)$$

where N and M are definite functions, given below.

Further, for the truncated torque integral,

$$\tau(B, r_0) = \int_{r_0}^{\infty} r dr \langle T_{\phi}^{\phi} \rangle ,$$

I write

$$2\pi \tau(B, r_0) = -3N(r_0) P_4(B) - M(r_0) P_2(B) . \quad (14)$$

Sometimes I will leave out the r_0 argument.

The first terms in (13) and (14) are those already discussed and, on their own, lead to the discrepancy, as will appear.

Algebra gives the intermediate manipulative equations

$$\begin{aligned} 2\pi \partial_{\beta} E(B) &= -B^2 \partial_B E(B) \\ &= -N B^2 \partial_B \frac{P_4(B)}{B} - M B^2 \partial_B \frac{P_2(B)}{B} \end{aligned}$$

with

$$-B^2 \partial_B \frac{P_n(B)}{B} = (1 - n) P_n - n$$

and so, from (13),

$$2\pi \partial_{\beta} E(B) = N(-3P_4 - 4) + M(-P_2 - 2) \quad (15)$$

whence, subtracting (14) from (15),

$$2\pi (\partial_{\beta} E(B) - \tau(B)) = -4N - 2M$$

whose required vanishing leads to the relation

$$M = -2N . \quad (16)$$

If M were zero, the situation is that of the previous section and (16) says that $N = 0$, which is the anomaly referred to because N is not zero, in fact the explicit formula for it is,

$$N(r_0) = \frac{2\pi}{720\pi^2} \int_{r_0}^{\infty} r dr \frac{1}{r^4} = -\frac{1}{720\pi} \frac{1}{r_0^2},$$

and so, from (16), the second, additional, term in the total energy, (13), is

$$\frac{1}{360\pi} \frac{1}{r_0^2} \frac{B^2 - 1}{B},$$

which I take to come from the pseudo-boundary at $r = r_0$ in the form of a localised extra contribution to the energy density,

$$\langle T_{00} \rangle^{extra} = \frac{B^2 - 1}{720\pi^2 r^3} \delta(r - r_0),$$

expressed in terms of the one-sided δ -function.

According to (14) there will also be an extra bit to the torque density,

$$\langle T_{\phi}^{\phi} \rangle^{extra} = -\langle T_{00} \rangle^{extra}. \quad (17)$$

5. Discussion

This does not represent a resolution of the difficulty since I cannot give a derivation of the additional pieces. I do not know how they might arise. In general form, they are similar to the extra terms described by Fursaev, [10], around a string, coming from a variational principle which allows for the cutoff radius, $r = r_0$, at which $\delta g_{\mu\nu}$ is non-vanishing. Unfortunately, the coefficients are not the same and, in fact, Fursaev's expressions satisfy (17) but with the opposite sign. (See also Saharian [11] for boundary effects.).⁷

I do believe, however, that the cavalier disposal of the apex divergence is part of the reason for the discrepancy and that the analysis of the annulus geometry, as promised in [1], or just that of an innerly truncated cone, will prove helpful.⁸

⁷ The variational approach in section 3 has to be modified for a smoothed, or cutoff, situation but I could not make the analysis consistent.

⁸ Incidentally, the series form of the Green function for the truncated cone, and related geometries, can be deduced from the list in Carslaw, [12].

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